Likelihood Ratio Tests

Recall the monotone likelihood ratio family.

Definition 1.1 *A family* $\mathfrak{F} = \{f(x,\theta) : \theta \in \Theta\}$ *is a* **monotone likelihood ratio family** *in the statistic* T *if* $f(x,\theta_0)/f(x,\theta_1)$ *is monotonically increasing in* T *for every* $\theta_0 > \theta_1$.

Remembering that power is a good thing, if we have two size α tests, the one with more power should be preferred.

Definition 1.2 *A test* φ *which is size* α *and which satisfies* $\mathrm{E}_\theta(\varphi(X)) \geq \mathrm{E}_\theta(\varphi^*(X))$ *for all* $\theta \in \Theta - \omega$ *(i.e.* θ *in the alternative) is called* **uniformly most powerful size** α *for* $H_0 : \theta \in \omega$ *versus* $H_1 : \theta \in \Theta - \omega$.

For monotone likelihood ratio families, these UMP tests can be found using the following theorem.

Theorem 1.1 *Consider* $\mathfrak{F} = \{f(x,\theta) : \theta \in \Theta\}$ *a monotone likelihood ratio family in* T*, and the hypothesis* $H_0 : \theta = \theta_0$ *versus* $H_1 : \theta > \theta_0$*. Then the uniformly most powerful test,* φ*, exists and is of the form*

$$\varphi(x) = \begin{cases} 1 & T(x) > c_\alpha \quad \textit{(reject)} \\ \gamma_\alpha & T(x) = c_\alpha \quad \textit{(randomize)} \\ 0 & T(x) < c_\alpha \end{cases} \tag{1}$$

where c_α *and* γ_α *are chosen to give size* α.

That this test is UMP can be seen via the following proof.

Proof 1.1 *Apply Neyman-Pearson to* $H_0 : \theta = \theta_0$ *versus* $H_1 : \theta = \theta_1$ *for* $\theta_1 > \theta_0$*. Then the Neyman-Pearson lemma indicates that the most powerful size* α *test (for this simple versus simple test) is*

$$\varphi(x) = \begin{cases} 1 & x \ni f_1(x)/f_0(x) > k_\alpha \\ \gamma_\alpha & \qquad\qquad\qquad = k_\alpha \\ 0 & \qquad\qquad\qquad < k_\alpha \end{cases} \tag{2}$$

But, f_1/f_0 *is monotonic in* T*, so* $f_1/f_0 > k_\alpha$ *if and only if* $T(x) > c_\alpha$ *for some* c_α*. Now,* c_α *is chosen under* H_0 *so that we get size* α*. So, the same test using* c_α *works for all* $\theta_1 > \theta_0$*. Hence,* $\varphi(X)$ *is UMP for* $H_0 : \theta = \theta_0$ *versus* $H_1 : \theta > \theta_0$.

Example 1.1 *Let* $X_1,\ X_2,\ldots,\ X_{200} \overset{iid}{\sim} N(\mu, \sigma_0^2)$. *Test the hypotheses*

$$H_0 : \mu = 100 \text{ versus } H_1 : \mu > 100$$

Suppose that $\alpha = 0.05$ *and that* $\sigma_0^2 = 2$. *Then*

$$\begin{aligned} \frac{f(\mathbf{x}, \mu_1)}{f(\mathbf{x}, \mu_0)} &= \exp\left(-\frac{\sum(x_i - \mu_1)^2}{4} + \frac{\sum(x_i - \mu_0)^2}{4}\right) && (3) \\ &= \exp\left(\frac{\mu_1 - \mu_0}{2}\sum x_i + \left(\mu_0^2 - \mu_1^2\right)\frac{200}{4}\right) && (4) \end{aligned}$$

for $\mu_1 > \mu_0$ *this is monotonic in* $\sum X_i$ *or* $\overline{X} = \sum X_i/n$. *So the UMP test rejects for large* $\overline{X}$.

Now, we need c_α *such that*

$$\begin{aligned} 0.05 &= \alpha && (5) \\ &= \mathrm{P}\left(\overline{X} > c_\alpha | \mu = \mu_0\right) && (6) \\ &= \mathrm{P}\left(\frac{\overline{X} - 100}{\sqrt{2}/\sqrt{200}} > \frac{c_\alpha - 100}{\sqrt{2}/\sqrt{200}}\right) && (7) \\ &= \mathrm{P}(Z > (c_\alpha - 100)10) && (8) \end{aligned}$$

So, $(c_\alpha - 100)10 = 1.645$ *or* $c_\alpha = 100.1645$ *and thus*

$$\begin{aligned} \varphi(\mathbf{x}) &= \begin{cases} 1 & \overline{x} > 100.1645 \\ 0 & \text{else} \end{cases} && (9) \\ &= \begin{cases} 1 & \sum x_i > 20032.9 \\ 0 & \text{else} \end{cases} && (10) \end{aligned}$$